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Post-Newtonian Cosmology

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Abstract

In cosmology, it is common to model the universe as being close to an exact solution of Einstein's equations, the homogeneous and isotropic Friedmann–Lemaître–Robertson–Walker spacetime. The inhomogeneities in the real universe that arise due to structure formation are modelled using perturbation theory in two different ways: cosmological perturbation theory and post-Newtonian theory. We review these two approaches and their assumptions and restrictions. Perturbation theory introduces a gauge freedom, but a recent work by Clifton, Gallagher, Goldberg, and Malik found that some of the well-studied gauge choices in cosmological perturbation theory are not viable in post-Newtonian theory. We discuss the gauge problem, review the paper of Clifton *et al.*, and extend its analysis to a new set of gauges, the Machian gauges of Bičák, Katz, and Lynden-Bell.

Contents

Chapter 1

Introduction

A principal goal of cosmology is to understand the properties and evolution of the largest structures in the universe. The largest level of structure is the cosmic web — voids that make up the majority of the volume of the universe, and sheets and filaments of galaxy clusters which surround and thread through these voids. To understand this structure, we must first study the early universe, devoid of large scale structure, and then understand and model the processes that have caused structure to form.

Approximately 380,000 years after the big bang, the primordial plasma making up the universe cooled to the point where protons and electrons could form hydrogen atoms, a period known as recombination. As hydrogen atoms are electrically neutral, this allowed photons to move through the universe mostly unimpeded compared to when the free electron density was much higher prior to recombination. Photon decoupling during this epoch produced primordial photons that formed the *cosmic microwave background* (CMB), and the surface we can trace these photons back to is known as the *surface of last scattering*.

Observations such as those from such as the Planck Collaboration [1] and the Wilkinson Microwave Anisotropy Probe (WMAP) [2] show that the CMB is almost perfectly isotropic. After subtracting a dipole anisotropy (believed to be due to the peculiar velocity of the Earth with respect to some cosmological frame), the CMB radiation differs from a perfect black-body spectrum by an amount on the order of one part in 100,000. If one assumes that this near-isotropy is true throughout the universe by the Copernican principle, then we expect that the universe was very close to being perfectly homogeneous during the epoch of last scattering. Due to the formation of structures, the universe is highly inhomogeneous on many scales during the current epoch. To understand and model the structures that exist in the universe today, it is important to be able to effectively model how small primordial fluctuations in the CMB (believed to be due to inflation of quantum fluctuations) have grown over the lifetime of the universe.

While the universe is not perfectly homogeneous, and can even be highly inhomogeneous on small scales, it is statistically homogeneous on scales significantly larger than the largest typical structures. This large-scale statistical homogeneity is essential to the standard ΛCDM model of cosmology. If the universe were perfectly homogeneous and isotropic, its geometry could be described by a Friedmann–Lemaître–Robertson–Walker (FLRW) metric [3], which can be derived from these assumptions. If the universe were *almost* perfectly homogeneous and isotropic, one might imagine that spacetime would be 'almost' FLRW.

With some additional assumptions about the matter content, Ellis and Stoeger [4] show that this is indeed the case. On account of this, it is standard to assume that on a large scale, the average geometry is described by Einstein's field equations with a spatially homogeneous perfect fluid as the matter source so that the universe's average cosmic evolution (for some suitable averaging procedure) is well described by the FLRW model. One should note, however, that these conditions assume that the geometry of the universe is described by an exact solution of Einstein's equations on any scale of averaging. The averaging problem entails many unresolved foundational problems [5–7] and it remains possible that the geometry evolves by a generic non-FLRW average if small scale Einstein equations which is statistically homogeneous in an appropriate sense.

Galaxy surveys such as the Sloan Digital Sky Survey show that at the present epoch the universe is dominated by voids, which make up the largest departures from spatial homogeneity. The largest typical voids have a mean effective diameter of $60h^{-1}\,\mathrm{Mpc}$ (the diameter of a sphere with the same volume as the void, as the void need not be spherical). Here h is the dimensionless Hubble parameter which is related to the Hubble constant by $H_0 = 100h\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, as this distance will depend on the experimental value of H_0. The size of these large voids restricts the scale on which the universe is statistical homogeneous to above approximately $100h^{-1}\,\mathrm{Mpc}$ in the current epoch. As the universe is increasingly homogeneous further back in its history, in earlier epochs this scale will be smaller.

One way of modelling an inhomogeneous universe is to assume that the average evolution is well described by the FLRW model (as the standard model does) and to account for inhomogeneities by adding small perturbations to this background geometry. This report considers two such approaches to this process in Chapter 3. Both of these are weak field expansions, so they require the gravitational potential to be small in order to have the geometry close to the FLRW geometry.

The first approach considered is cosmological perturbation theory (Section 3.3), which is only valid when the deviations of various geometric and matter quantities are sufficiently small compared to the background FLRW spacetime. There are no fundamental restrictions on the scales on which cosmological perturbation theory can be applied as long as all relevant quantities are small. In the early universe this was the case on all scales, so cosmological perturbation theory is widely applicable during these early epochs, but the emergence of structure means that the scale of the system being studied must be larger than the scale of statistical homogeneity for the assumptions of cosmological perturbation theory to hold. This corresponds to scales of over $100h^{-1}\,\mathrm{Mpc}$ in the current epoch. This report only discusses first-order perturbations, however considering second-order (or higher) perturbations can give nonlinear corrections [8].

The second approach is post-Newtonian theory, discussed in Section 3.4. The main advantage of post-Newtonian theory is that it is applicable even in the presence of arbitrarily large density contrasts (which prevent the application of cosmological perturbation theory), making it more useful for describing the nonlinear structures that occur on small scales. The assumptions of post-Newtonian theory place limits on the scale on which it is applicable, because they require that the background geometry changes by only a small amount in the time it takes information to propagate across the system. It is possible to combine these two approaches into a two-parameter perturbation theory that can be used in the presence of nonlinear relativistic gravity while also including the large scale fluctuations of cosmological perturbation theory [9, 10].

On small scales, the nonlinearity in the matter distribution causes the assumptions underlying weak field expansions to break down so these linear approximations cannot be used. Instead, numerical simulations are used. Typically N-body simulations are used to simulate structure formation, and usually Newtonian gravity is used as an approximation since simulating general relativity is not only more computationally intensive but also requires resolving fundamental ambiguities in splitting space and time. As cosmological observations become more precise, increased precision in simulations of structure formation is also desired, so it is becoming increasingly necessary to consider simulations that work with the full theory of general relativity. These numerical relativity simulations [11–14] are important to determine whether there are any significant relativistic effects that are neglected by using the standard model of cosmology or Newtonian N-body simulations. Post-Newtonian theory is useful for understanding how relativistic effects might impact precision observations and change the results of Newtonian simulations.

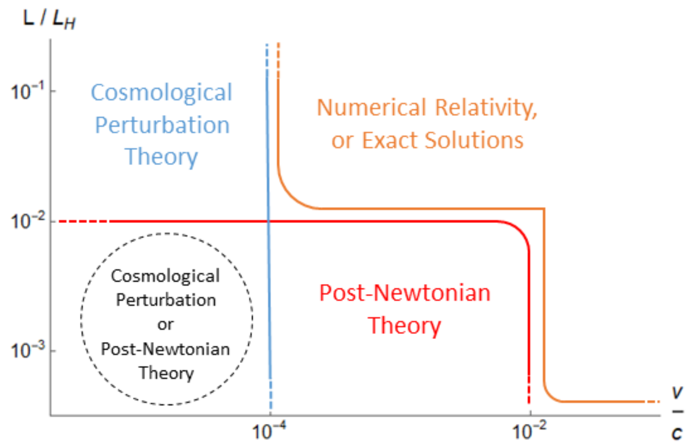

Figure 1.1: The scales and velocities for which each approach is applicable. The horizontal axis gives peculiar velocity of matter sources, and the vertical axis the scale of the system as a fraction of the Hubble scale $L_H = c/H_0$. This figure is reproduced from [15].

1.1 The Gauge Problem

Perturbation theory considers two distinct manifolds: one describing the ficticious background spacetime (in this report, FLRW spacetime), and a second to model the real inhomogeneous universe, which we construct by adding a small perturbation to the background spacetime. However, for a given perturbed spacetime, the choice of background and perturbation is not unique; there may be different coordinate systems for the same perturbed spacetime that have a FLRW background plus some different perturbation. The freedom to pick a different background and perturbation to represent the same physical situation is called a *gauge freedom*, and transformations between these coordinate systems are called *gauge transformations*, which are the subject of Chapter 4. There are several equivalent ways to view gauge transformations. Formally, a gauge transformation can be thought of as changing the diffeomorphism that maps points in the background to points in the perturbed spacetime. From this perspective, the gauge freedom arises from the fact that there are many possible diffeomorphisms between the background and perturbed spacetimes. Less formally, a gauge transformation can be thought of as an infinitesimal coordinate transformation.

Gauge transformations are distinct from regular coordinate transformations because as well as inducing a coordinate change on the physical perturbed spacetime, a gauge transformation changes the point in the background spacetime that corresponds to a given point in the perturbed spacetime. This means that even if a quantity is a scalar and so will not change under a coordinate transformation, the part of that scalar that is considered to be due to the perturbation may not be invariant under a gauge transformation because the associated point in the background spacetime has changed.

Gauge freedom is useful because it allows one to simplify problems by making a gauge transformation to put the metric into a form more suitable to the problem at hand. One example of this is the synchronous gauge, which is equivalent to picking Gaussian normal coordinates and puts the metric into a form in which the perturbations to its time-time component and time-space components vanish, simplifying many calculations. Gauge choices are often adapted to foliations of spacetime by hypersurfaces that have desirable properties (as discussed in Section 2.2). The gauges in [16] apply to hypersurfaces with various notions of spatial homogeneity, such as defining hypersurfaces where the expansion rate within hypersurfaces is uniform. Bičák, Katz, and Lynden-Bell [17] consider other possible foliations that give gauge choices that best embody Mach's principle.

Since coordinate systems related by a gauge transformation all represent the same physical situation, what happens in the universe should be independent of how the split into background and perturbation is done. However, if the gauge condition chosen leaves residual degrees of gauge freedom there can be spurious gauge mode solutions to the perturbation equations that have no physical meaning. Hawking [18] made an early attempt to formulate the perturbation equations in a completely covariant form to avoid this, but this approach was flawed in that the necessary hypersurfaces could not be constructed in the presence of pressure perturbations [16]. This problem motivates finding quantities that are gauge invariant, as was done by Bardeen [16].

1.2 Mach's Principle

There are many different principles that take the name "Mach's principle" (Bondi and Samuel [19] list 10), but all of these have properties fundamentally described by the following statement: *"The universe, as represented by the average motion of distant galaxies does not appear to rotate relative to local inertial frames"*[19]. Bičák, Katz, and Lynden-Bell [20] give a detailed overview of Mach's principle, adapt Bondi's original formulation: *"Local inertial frames are determined through the distributions of energy and momentum in the Universe by some weighted averages of the apparent motions"* [21]. In [17], Bičák *et al.* use this to determine what data are needed to determine accelerations and rotations of local frames (relative to a frame of "cosmological observers"), and use this to define gauges choices that embody their formulation of Mach's principle. In these gauges, the distribution of energy and momentum (through the energy-momentum tensor) can be used to uniquely determine the instantaneous rotations and accelerations of local inertial frames.

1.3 Outline

Chapter 2 reviews the $1+3$ (threading) and $3+1$ (slicing) descriptions of spacetime and introduces geometric properties of hypersurfaces. Chapter 3 discusses two approaches to weak field expansions about a FLRW background: cosmological perturbation theory in Section 3.3, and post-Newtonian theory in Section 3.4, in which the expression for extrinsic curvature in post-Newtonian theory is derived, to be used for the study of the Mach 1 gauge. Chapter 4 discusses gauge transformations in the context of these two weak field expansions, and derives how metric perturbations change under a gauge transformation to arrive at the results previously achieved in [15]. It then reviews the analysis of the viability of two gauges previously studied in [15] and then discusses the Machian gauges introduced in [17]. These three gauges have not been previously studied in the context of post-Newtonian theory, and the viability of each of these gauges in post-Newtonian theory is determined. Chapter 5 is a brief concluding discussion.

My own contributions to this project involved reproducing the calculations of Clifton *et al.* in [15] outlined in Sections 4.1 and 4.2, and then applying these methods to the gauges proposed by Bičák, Katz, and Lynden-Bell. In particular, Sections 4.3.1–4.3.3 which establish the post-Newtonian equivalents of the Mach 1, Mach 2, and Mach 3 gauges are all original work. Similarly, Section 3.4.1 and Appendix A contain results which have not been found in the literature.

Chapter 2

Threading and Slicing

One way of describing spacetime is through congruences of non-intersecting timelike curves. This approach is known as "threading", or the $1+3$ formalism. If the manifold is globally hyperbolic (which will be assumed to be the case), spacetime can be foliated into hypersurfaces [22] ("slicing", the $3+1$ formalism). These two views are related, as congruences of curves that are vorticity-free can be used to construct hypersurfaces that they are orthogonal to.

2.1 Congruences of Curves

Spacetime is modelled as a globally hyperbolic four-dimensional pseudo-Riemannian manifold, which can be "threaded" by a congruence of non-intersecting timelike curves using coordinates (p, x^i), where fixed x^i defines one of the timelike curves, and p is a parameter along this curve. These timelike curves can be thought of as the worldlines of "cosmological observers", where p would typically be cosmic time t or the observers' proper time. Let u^μ be the normalised ($u^\mu u_\mu = -1$, in units where $c = 1$) vector along the curves in the congruence (that is, the normalised velocity vector of the cosmological observers). For a congruence of timelike curves with a velocity vector u^μ a *projection tensor* $b_{\mu\nu}$ can be defined:

$$b_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu. \tag{2.1}$$

As the name suggests, this tensor will project a tensor into the hyperplane that is locally orthogonal to the congruence.

It is standard (see e.g. [23]) to decompose the covariant derivative (denoted by a semicolon followed by one or more indices) of a 4-velocity into components that show geometric properties of the congruence of curves:

$$\nabla_\mu u_\nu \equiv u_{\nu;\mu} = -u_\mu \alpha_\nu + \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\theta b_{\mu\nu}. \tag{2.2}$$

Here α_μ is the acceleration, and $\omega_{\mu\nu}$ is the vorticity, which is antisymmetric. The remaining symmetric part is split into the shear $\sigma_{\mu\nu}$, which is the traceless symmetric part, and a trace term proportional to the expansion θ. These components are given by

$$\alpha_\mu = u_{\mu;\nu} u^\nu \tag{2.3}$$
$$\omega_{\mu\nu} = b^\rho{}_\mu b^\sigma{}_\nu u_{[\rho;\sigma]} \tag{2.4}$$
$$\sigma_{\mu\nu} = b^\rho{}_\mu b^\sigma{}_\nu u_{(\rho;\sigma)} - \frac{1}{3}\theta b_{\mu\nu} \tag{2.5}$$
$$\theta = u^\mu{}_{;\mu} \tag{2.6}$$

where $u_{[\rho;\sigma]} = \frac{1}{2}(u_{\rho;\sigma} - u_{\sigma;\rho})$ is the antisymmetric part of $u_{\rho;\sigma}$, and $u_{(\rho;\sigma)} = \frac{1}{2}(u_{\rho;\sigma} + u_{\sigma;\rho})$ is the symmetric part.

2.2 Foliations

An alternative perspective is to look at spacetime as being foliated by a one-parameter family of spacelike hypersurfaces, which are given as level sets of the parameter t. Global hypersurfaces can only be formed from congruences of timelike curves that are vorticity-free. In this case, the hyperplanes orthogonal to a local part of the congruence can be joined together to form hypersurfaces that foliate spacetime, which is not possible in the presence of vorticity. As the congruences of curves describing the motion of bodies such as stars in galaxies or galaxies in galaxy clusters have vorticity, in practice this means that averaging over small-scale vorticity is needed to construct an average cosmic fluid free of vorticity which can be used to construct hypersurfaces. No general procedure for doing this is known, making the “fitting problem” an outstanding foundational problem in cosmology [5, 6].

The normal vector field n^μ to these global hypersurfaces is taken to be timelike and normal ($g_{\mu\nu} n^\mu n^\nu = -1$), and it need not coincide with the fluid 4-velocity u^μ, which may have vorticity. Each hypersurface is given a time coordinate t which is constant on the hypersurface, and increasing along the flow of n^μ. The *lapse* function N, positive at every point, determines how far apart hypersurfaces are when moving in the direction of the normal vector n^μ. This description gives the one-form n_μ the following form in terms of the lapse:

$$n_\mu = -N(1, 0). \tag{2.7}$$

The *shift vector* N^μ determines how the spatial coordinates, which have been kept arbitrary, change between slices. The unit vector $\boldsymbol{\partial}_t = (1, 0, 0, 0)$ determining the direction forward in time at a constant spatial position can then be written as a function of the lapse, normal vector, and shift,

$$\boldsymbol{\partial}_t = N\boldsymbol{n} + \boldsymbol{N}. \tag{2.8}$$

Choices of foliation such as those given in Section 4.3 amount to setting the shift and lapse. From the above equation an expression for the normal vector can be obtained, giving

$$n^\mu = \frac{1}{N}(1, -N^i). \tag{2.9}$$

The fluid 4-velocity u^μ, which is in general tilted with respect to n^μ, can be decomposed as

$$u^\mu = \gamma(n^\mu + v^\mu) \tag{2.10}$$

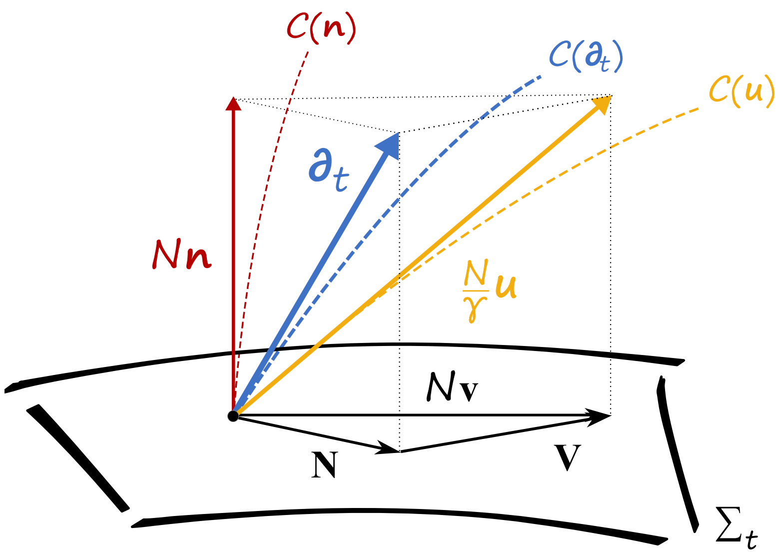

Figure 2.1: A spatial hypersurface Σ_t showing the vector fields discussed in this section. $N\boldsymbol{n}$ is normal vector scaled by the lapse, tangent to a congruence $C(\boldsymbol{n})$; $\boldsymbol{\partial}_t$ is the time vector of the coordinate basis, tangent to the congruence of curves $C(\boldsymbol{\partial}_t)$ with x^i = constant; and $\boldsymbol{u}$ is the 4-velocity of the fluid, tangent to the congruence $C(\boldsymbol{u})$. The relationships in Eq. (2.8), Eq. (2.10) and Eq. (2.11) can be seen in this figure. This figure is reproduced from [24].

where γ is the Lorentz gamma factor and v^μ is the velocity of the fluid in the hypersurface, (in a frame at rest in the hypersurfaces and transported along the normal vector)[1], called the *Eulerian velocity*, which determines the aforementioned tilt between n^μ and u^μ. For the later sections of this report, the fluid will be taken to be hypersurface orthogonal, so this tilt will vanish, but it is useful to consider tilt for the construction of different spatial hypersurfaces to which the fluid is not necessarily hypersurface orthogonal, even if it has no vorticity. Defining $\boldsymbol{V}$ as the spatial coordinate velocity of the fluid, tangent to the hypersurfaces, allows the Eulerian velocity to be written as

$$v^\mu = \frac{1}{N}(N^\mu + V^\mu). \tag{2.11}$$

These vectors are illustrated in figure 2.1.

Another projection tensor, $P_{\mu\nu}$, this time to project into the global hypersurface, can be defined as

$$P_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{2.12}$$

In general, $P_{\mu\nu}$ and $b_{\mu\nu}$ will differ because of the tilt between n^μ and u^μ.

The *extrinsic curvature* $K_{\mu\nu}$ of a spatial hypersurface with normal vector n^μ is defined as

$$\begin{aligned} K_{\mu\nu} &= \frac{1}{2}\mathcal{L}_{\boldsymbol{n}} P_{\mu\nu} && (2.13)\\ &= \nabla_\mu n_\nu + n_\mu n^\lambda n_{\nu;\lambda}, && (2.14)\end{aligned}$$

where $\mathcal{L}_{\boldsymbol{n}}$ is the Lie derivative along the normal vector field. The extrinsic curvature describes the curvature of the hypersurfaces when embedded in the full spacetime manifold. It is

[1] We will use v to denote the peculiar velocity of the fluid with respect to the average perfect fluid velocity, but no confusion will arise because outside of this section the v here will be zero.

distinct from the intrinsic curvature, which is measured by the Riemann tensor determined from the intrinsic 3-dimensional metric defined by $P_{\mu\nu}$. As an example of the difference, consider a two-torus embedded in three-dimensional Euclidean space. The torus on its own can be equipped with a flat two-dimensional metric lacking any intrinsic curvature, but a torus embedded in three-dimensional space will always be curved within that space, having nonzero extrinsic curvature. From Eq. (2.13) it can be seen that $K_{\mu\nu}$ is symmetric (as $P_{\mu\nu}$ is symmetric).

The extrinsic curvature can be related to the geometric properties of the congruence of timelike curves that have the normals to the hypersurfaces as their "velocity" vectors. Notice that for such a congruence, the acceleration takes the form $\alpha_\nu = n^\lambda n_{\nu;\lambda}$, so Eq. (2.14) becomes

$$K_{\mu\nu} = n_{\nu;\mu} + n_\mu \alpha_\nu. \tag{2.15}$$

Expanding $n_{\nu;\mu}$ as in Eqs. (2.2)–(2.6), and using the fact that $K_{\mu\nu}$ is symmetric, we find

$$K_{\mu\nu} = \sigma_{\mu\nu} + \frac{1}{3}\theta P_{\mu\nu}. \tag{2.16}$$

Note that the projectors $b_{\mu\nu}$ and $P_{\mu\nu}$ coincide because the 4-velocity of the congruence is hypersurface orthogonal. The trace of the extrinsic curvature $\mathcal{K}$, sometimes called the scalar extrinsic curvature, is

$$\mathcal{K} = g^{\mu\nu} K_{\mu\nu} = \nabla_\mu n^\mu. \tag{2.17}$$

Notably, this means that for a congruence of timelike curves orthogonal to the hypersurfaces, the scalar extrinsic curvature is equal to the expansion θ of the congruence. Eq. (2.17) can be expanded to give

$$\begin{aligned} \nabla_\mu n^\mu &= \partial_\mu n^\mu + \Gamma^\mu{}_{\mu\nu} n^\nu & (2.18) \\ &= \partial_0 n^0 + \partial_k n^k + \Gamma^k{}_{kj} n^j + \Gamma^0{}_{00} n^0 + \Gamma^k{}_{k0} n^0 + \Gamma^0{}_{0k} n^k. & (2.19) \end{aligned}$$

which will be used later to relate metric perturbations to perturbations in the extrinsic curvature.

The notion of extrinsic curvature and other geometric properties of foliations provide the motivation for certain gauge choices, described in Section 4.3, in which the slices have useful geometric properties, such as constant extrinsic or intrinsic scalar curvature. Several such gauge conditions were considered by Bardeen [16], including the minimal shear hypersurface condition and the uniform Hubble expansion gauge, both discussed in [17] and in Section 4.3.

Chapter 3

Weak Field Expansions

3.1 The Background FLRW Spacetime

If the universe is assumed to be perfectly homogeneous and isotropic, and expanding over time by an equal amount at each point in space, then the geometry of this spacetime can be described by the Friedmann–Lemaître–Robertson–Walker (FLRW) metric $\bar{g}_{\mu\nu}$ with line element

$$\bar{g}_{\mu\nu}dx^{\mu}dx^{\nu} = a^2(-d\tau^2 + f_{ij}dx^i dx^j), \tag{3.1}$$

where $a(\tau)$ is the scale factor and τ is conformal time

$$a\, d\tau = dt. \tag{3.2}$$

Throughout this report, the background spatial metric f_{ij} will be assumed to be flat, i.e., $f_{ij} = \delta_{ij}$ with curvature index $k = 0$. Small amounts of curvature can be introduced through perturbations described in the next section. This is a standard assumption in cosmology, justified by measurements such as those from the Planck Collaboration [1], which found that $\Omega_k = 0.001 \pm 0.002$. Recently, some have claimed evidence for a somewhat larger spatial curvature based on different analysis of Planck data [25], but this claim is disputed [26]. For an overview of cosmological perturbation theory that analyses the cases of $k = \pm 1$ in addition to $k = 0$ as done here, see [17].

It will also be assumed that the matter content of spacetime can be modelled by a perfect fluid, with energy-momentum tensor

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)u^{\mu}u^{\nu} + pg^{\mu\nu}. \tag{3.3}$$

Several important equations can be derived by studying Einstein's field equations with this perfect fluid energy-momentum tensor and the background FLRW geometry. The background Christoffel symbols can be computed (see e.g. [27]), to give

$$\bar{\Gamma}^0{}_{00} = \mathcal{H}, \quad \bar{\Gamma}^0{}_{ij} = \mathcal{H}\delta_{ij}, \quad \bar{\Gamma}^i{}_{0j} = \mathcal{H}\delta^i_j. \tag{3.4}$$

All other Christoffel symbols are zero because the metric is spatially flat. We introduce the conformal Hubble parameter $\mathcal{H} = a'/a$ and denote a derivative with respect to τ with a prime. Using these Christoffel symbols, the non-vanishing components of the Ricci tensor are

$$\bar{R}_{00} = -3(\mathcal{H}' - \mathcal{H}^2), \quad \bar{R}_{ij} = (\mathcal{H}' + \mathcal{H}^2)\delta_{ij}. \tag{3.5}$$

Using the above, the time-time component of Einstein's field equations is

$$\mathcal{H}^2 = \frac{8\pi G}{3}\bar{\rho}a^2 + \frac{1}{3}\Lambda a^2 \tag{3.6}$$

and the space-space components give

$$\mathcal{H}' = -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{p})a^2 + \frac{1}{3}\Lambda a^2. \tag{3.7}$$

These two equations are the Friedmann equations. The time-space components give the energy-momentum conservation equation (equivalent to $\nabla_\mu T^{\mu\nu} = 0$),

$$\bar{\rho}' + 3\mathcal{H}(\bar{\rho} + \bar{p}) = 0. \tag{3.8}$$

The acceleration, shear, and vorticity of the perfect fluid (see Section 2.1) vanish in the FLRW spacetime, leaving only the expansion, which is $\bar{\theta} = 3\mathcal{H}/a$.

3.2 Perturbation Theory

To account for small inhomogeneities in the geometry of spacetime resulting from the presence of an inhomogeneous distribution of matter (large-scale structure such as galaxy clusters and voids) a small perturbation can be added to the background metric. What it means for a perturbation to be "small" is the subject of sections 3.3 and 3.4.

The perturbed metric is denoted

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{3.9}$$

Similarly, for some $\delta g^{\mu\nu}$,

$$g^{\mu\nu} = \bar{g}^{\mu\nu} + \delta g^{\mu\nu}. \tag{3.10}$$

Since $g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu{}_\nu$, this $\delta g^{\mu\nu}$ can be computed in terms of $h_{\mu\nu}$.

$$\begin{aligned}
\delta^\mu{}_\nu &= g^{\mu\lambda}g_{\lambda\nu} && (3.11)\\
&= \left(\bar{g}^{\mu\lambda} + \delta g^{\mu\lambda}\right)(\bar{g}_{\lambda\nu} + h_{\lambda\nu}) && (3.12)\\
&= \delta^\mu{}_\nu + \bar{g}_{\lambda\nu}\delta g^{\mu\lambda} + \bar{g}^{\mu\lambda}h_{\lambda\nu} + O(h^2) && (3.13)\\
\bar{g}_{\lambda\nu}\delta g^{\mu\lambda} &\approx -\bar{g}^{\mu\lambda}h_{\lambda\nu} && (3.14)\\
\delta g^{\mu\nu} &\approx -h^{\mu\nu} && (3.15)
\end{aligned}$$

So

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu}, \tag{3.16}$$

where $h^{\mu\nu}$ has been raised with the background metric.

Using the Helmholtz theorem, the perturbations $h_{\mu\nu}$ can be decomposed into

$$h_{00} = -2a^2\phi \tag{3.17}$$
$$h_{0i} = a^2(B_{,i} - S_i) \tag{3.18}$$
$$h_{ij} = a^2(-2\psi\delta_{ij} + 2E_{,ij} + 2F_{(i,j)} + D_{ij}) \tag{3.19}$$

where S_i and F_i are divergenceless vector field components (that is, $S^i{}_{,i} = 0 = F^i{}_{,i}$) and D_{ij} is traceless and divergenceless ($D^i{}_i = 0$, $D^i{}_{j,i} = 0$).

Two weak field expansions are discussed in the following sections: first, cosmological perturbation theory, which is valid when peculiar velocities and density contrasts remain small; then post-Newtonian theory, which has no requirement for small density contrasts but is only valid on small spatial scales. For weak field expansions like these, the gravitational potential ϕ typically has magnitude $\lesssim 10^{-4}$.

3.3 Cosmological Perturbation Theory

In cosmological perturbation theory, every perturbative quantity has its order-of-smallness quantified in terms of a small parameter ε, such that

$$v \sim \frac{\delta\rho}{\rho} \sim \phi \sim B_{,i} \sim S_i \sim \psi \sim F_{i,j} \sim E_{,ij} \sim D_{ij} \sim \varepsilon \ll 1 \tag{3.20}$$

where v is the peculiar velocity of matter fields with respect to the "average" perfect fluid flow, and $\delta\rho/\rho$ is the density contrast. The remaining quantities are the components of the metric perturbation given in Eqs. (3.17)–(3.19). That these quantities are required to be small is a consequence of assuming that the real universe differs only by a small amount from the background FLRW model. As discussed in the introduction, this restricts the applicability of cosmological perturbation theory to epochs and scales where this assumption can be justified. This means that the scale of the system being modelled must be one on which the universe appears statistically homogeneous, which limits cosmological perturbation theory to scales above approximately $100h^{-1}$ Mpc in the current epoch. As the universe was more homogeneous earlier in its evolution, cosmological perturbation theory can be used on smaller scales in the early universe.

Cosmological perturbation theory includes assumes that multiplying a quantity by background quantities such as the conformal Hubble parameter $\mathcal{H} = a'/a$ or the background density $\bar{\rho}$ does not change its order of magnitude, nor does taking the derivative of a quantity.

For a more complete review of cosmological perturbation theory, see e.g. [28] or relevant chapters of [27].

3.4 Post-Newtonian Theory

The post-Newtonian expansion is a slow-motion expansion which, unlike standard cosmological perturbation theory, remains valid when density contrasts are large. This report follows the treatment of post-Newtonian theory in [15]. The key aspect of post-Newtonian theory is that it is restricted to scales small enough that the change in the background geometry over time is negligible in the time it takes information to propagate from one side of the system under consideration to the other.

To formalise this condition, let τ_c be the characteristic (conformal) time scale for variations of the background geometry. Then the characteristic length scale of the system $\lambda_c = c\tau_c$ gives the approximate distance that light can travel over the time scale for these variations. Denote the spatial scale of the system as r_c. The assumption for the post-Newtonian expansion that distance scales in the system are small compared to the characteristic length scale determined by dynamical changes of the background due to gravity can now be written as $r_c \ll \lambda_c$. As typical 3-velocities of matter v_c will be of order

$$v_c = \frac{r_c}{\tau_c}, \tag{3.21}$$

this assumption that $r_c \ll \lambda_c$ implies that $v_c \ll c$, so this can also be seen as a slow-motion condition.

State variables such as pressure and energy density will also change slowly over time, as they are primarily due to the motion of the matter sources. Additionally, because the source of the metric perturbations is the energy-momentum tensor, the metric perturbations should similarly vary only slowly over time. This gives the general rule that for any quantity A,

$$\left|\frac{\partial A/\partial \tau}{\partial A/\partial x}\right| \sim \left|\frac{v_c}{c}\right|. \tag{3.22}$$

Since $v_c \ll c$, this implies that time derivatives of quantities are much smaller than spatial derivatives of those quantities. This motivates defining

$$\eta = \frac{v_c}{c} \ll 1 \tag{3.23}$$

as an order-of-smallness parameter for post-Newtonian theory. The orders of magnitude of post-Newtonian quantities such as the components of the metric perturbation can be found in terms of η.

The matter content is modelled as a single perfect fluid, so the energy-momentum tensor is given by

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)u^\mu u^\nu + pg^{\mu\nu}. \tag{3.24}$$

Since $v^i \ll c$, the 4-velocity of the fluid can be approximated as $u^\mu \approx (c, v^i)$. This gives the components of $T^{\mu\nu}$ as:

$$T^{00} \approx \rho c^2, \tag{3.25}$$

$$T^{0i} \approx \rho v^i c, \tag{3.26}$$

$$T^{ij} \approx \rho v^i v^j + p\delta^{ij}. \tag{3.27}$$

The relative magnitudes of the components are

$$\frac{T^{0i}}{T^{00}} \sim \frac{v_c}{c} \qquad \frac{T^{ij}}{T^{00}} \sim \frac{v_c^2}{c^2}, \tag{3.28}$$

that is, $T^{ij} \ll T^{0i} \ll T^{00}$. From the field equations, the above equations imply that

$$\frac{h_{0i}}{h_{00}} \sim \frac{v_c}{c}, \qquad \frac{h_{ij}}{h_{00}} \sim 1. \tag{3.29}$$

The virial relation $\phi \sim \frac{v^2}{c^2}$ gives $h_{00} \sim \eta^2$, so from Eq. (3.29), $h_{0i} \sim \eta^3$ and $h_{ij} \sim \eta^2$. Since $\phi \sim 10^{-4}$ empirically for systems of interest, the virial relation also tells us that $\eta \sim \frac{v}{c} \sim 10^{-2}$. The virial theorem is applicable here because it is derived from the Newtonian limit of Einstein's field equations, so it is thought to be a good approximation in most nonlinear situations where ϕ and v are small.

The quantities in the Helmholtz expansion from Eqs. (3.17)–(3.19) must be of the same orders of magnitude as the corresponding metric perturbation, so

$$\phi \sim \eta^2 \tag{3.30}$$

$$B_{,i} \sim S_i \sim \eta^3 \tag{3.31}$$

$$\psi \sim E_{,ij} \sim F_i \sim D_{ij} \sim \eta^2 \tag{3.32}$$

3.4.1 Extrinsic Curvature in Post-Newtonian Theory

In Section 2.2 it was noted that for a spatial slicing of spacetime into $\tau =$ constant hypersurfaces, the expansion θ of a congruence of timelike curves orthogonal to the slices gave a geometric quantity $\mathcal{K}$, the extrinsic curvature scalar.

Since the fluid velocity is hypersurface orthogonal the fluid 4-velocity $u^\mu = a^{-1}(1 - \phi + \frac{v^2}{2}, v^i)$ can be used as the normal vector. Substituting this into Eq. (2.19) gives

$$\nabla_\mu n^\mu = \partial_0 u^0 + \partial_k u^k + \Gamma^k{}_{kj} u^j + \Gamma^0{}_{00} u^0 + \Gamma^k{}_{k0} u^0 + \Gamma^0{}_{0k} u^k. \tag{3.33}$$

Term by term, using the Christoffel symbols determined in Appendix A, including terms of up to order η^4,

$$\partial_0 u^0 = \frac{1}{a}\big(\mathcal{H}u^0 - \phi' + {v_k}' v^k\big) \tag{3.34}$$

$$\partial_k u^k = \frac{1}{a}\partial_k v^k \tag{3.35}$$

$$\Gamma^0{}_{00} u^0 = \frac{1}{a}\big(\phi' + \mathcal{H}u^0\big) \tag{3.36}$$

$$\Gamma^0{}_{0k} u^k = \frac{1}{a}\phi_k v^k \tag{3.37}$$

$$\Gamma^k{}_{k0} u^0 = \frac{1}{a}\big(-3\mathcal{H}u^0 + \partial_0(3\psi - \nabla^2 E)\big) \tag{3.38}$$

$$\Gamma^k{}_{kj} u^j = -\frac{1}{a} v^k \partial_k\big(3\psi - \nabla^2 E\big) \tag{3.39}$$

Combining the above,

$$\begin{aligned}\theta &= \nabla_\mu n^\mu \\ &= \frac{1}{a}\left(\partial_k v^k + 3\mathcal{H}\left(1 - \phi + \frac{v^2}{2}\right) + \phi_{,k} v^k + {v_k}' v^k\right) - u^\mu \partial_\mu\big(3\psi - \nabla^2 E\big) + O(\eta^5).\end{aligned} \tag{3.40}$$

Chapter 4

Gauge Transformations

As discussed in Chapter 1, in perturbation theory there is a gauge freedom to make certain changes of coordinates that may simplify the mathematics of calculations but do not cause any change in physical meaning. This chapter discusses these transformations in the context of cosmological perturbation theory and post-Newtonian theory and investigates several gauges. Because of the differences in the magnitude of quantities in post-Newtonian theory compared to cosmological perturbation theory, many gauges that are commonly used in cosmological perturbation theory are not viable in post-Newtonian theory, as a recent paper [15] found. Section 4.3 investigates the viability of another set of gauges in post-Newtonian theory.

Suppose a coordinate transformation transforms the metric to

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \Delta g_{\mu\nu}. \tag{4.1}$$

If $\Delta g_{\mu\nu}$ is small enough (of the same order as $h_{\mu\nu}$, or smaller), this change can be thought of as a change to only the field perturbations, and not to the background, such that

$$\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \left(h_{\mu\nu} + \Delta g_{\mu\nu}\right) = \bar{g}_{\mu\nu} + \tilde{h}_{\mu\nu}, \tag{4.2}$$

where $\tilde{h}_{\mu\nu}$ is some different perturbation. Here, the resulting metric is also a perturbation of the background metric with the same constraints on the magnitude of the perturbation, that is, it remains "close" to the background FLRW metric (for the given weak field expansion's definition of what being close to the background means). It is important that the new perturbation remains small to avoid violating any of the assumptions of the weak field expansion. Such a transformation is called a *gauge transformation.* A gauge transformation can be written in terms of how it changes the coordinates,

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x). \tag{4.3}$$

The vector ξ^μ is called the *gauge generator* for this transformation. By studying the effect of a gauge transformation on various quantities such as the metric, the constraints on the magnitude of the gauge generator can be determined for each of the two weak-field expansions.

4.1 Magnitude of Gauge Transformations

Under a gauge transformation with generator ξ^μ, a tensor field Q changes as

$$\tilde{Q}^{\alpha_1\alpha_2\cdots}{}_{\beta_1\beta_2\cdots} = \exp\big(\mathcal{L}_\xi\big) Q^{\alpha_1\alpha_2\cdots}{}_{\beta_1\beta_2\cdots} \tag{4.4}$$

In particular, the transformed metric becomes

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu} + \frac{1}{2}\mathcal{L}_\xi^2 g_{\mu\nu} + \ldots \tag{4.5}$$

so, omitting terms at quadratic order or higher in ξ,

$$\tilde{g}_{00} = g_{00} + g_{00,\mu}\xi^\mu + 2g_{0\mu}\xi'^\mu + \cdots \tag{4.6}$$
$$\tilde{g}_{0i} = g_{0i} + g_{0i,\mu}\xi^\mu + g_{0\mu}\xi^\mu{}_{,i} + g_{i\mu}\xi'^\mu + \cdots \tag{4.7}$$
$$\tilde{g}_{ij} = g_{ij} + g_{ij,\mu}\xi^\mu + 2g_{\mu(i}\xi^\mu{}_{,j)} + \cdots \tag{4.8}$$

This places some constraints on the size of ξ^μ, because as discussed at the start of the section, the perturbation part of this transformed metric must meet the same size constraints as the original perturbation. In cosmological perturbation theory $\tilde{h}_{\mu\nu} \sim \varepsilon$, so this means that $\xi^\mu \sim \varepsilon$.

In post-Newtonian theory, the components of the metric are not all of the same order of magnitude, as shown in Eq. (3.29). By Eq. (3.31), the perturbation part of $\tilde{g}_{0i}$ must be of order η^3, so Eq. (4.7) gives

$$\tilde{g}_{0i} = g_{0i} + g_{0i,\mu}\xi^\mu + g_{0\mu}\xi^\mu{}_{,i} + g_{i\mu}\xi'^\mu \sim \eta^3. \tag{4.9}$$

Looking at the term $g_{0\mu}\xi^\mu{}_{,i}$, and recalling that the background metric is diagonal, gives the constraint

$$g_{00}\xi^0{}_{,i} \sim \eta^3. \tag{4.10}$$

This restricts the size of ξ^0 to at most $O(\eta^3)$. Similarly, Eq. (3.32) and Eq. (4.8) give

$$\tilde{g}_{ij} = g_{ij} + g_{ij,\mu}\xi^\mu + 2g_{\mu(i}\xi^\mu{}_{,j)} \sim \eta^2. \tag{4.11}$$

This condition shows that $\xi^i \sim \eta^2$. Recall that taking the time derivative of a quantity makes it smaller by an order of η, so the above equation still agrees with the previous constraint $\xi^0 \sim \eta^3$. Eq. (4.6) does not tighten the constraints on the size of the components of ξ any further.

To summarise, the above equations give conditions on the order of smallness of the components of an arbitrary gauge generator ξ^μ:

$$\xi^0 \sim \eta^3, \tag{4.12}$$
$$\xi^i \sim \eta^2. \tag{4.13}$$

and so in post-Newtonian theory, the leading-order terms of Eqs. (4.6)–(4.8) are

$$\tilde{g}_{00} = g_{00} + g_{00}{}'\xi^0 + g_{00,i}\xi^i + 2g_{00}\xi'^0 + O(\eta^5), \tag{4.14}$$
$$\tilde{g}_{0i} = g_{0i} + g_{00}\xi^0{}_{,i} + g_{ij}\xi'^j + O(\eta^4), \tag{4.15}$$
$$\tilde{g}_{ij} = g_{ij} + 2g_{k(i}\xi^k{}_{,j)} + O(\eta^3). \tag{4.16}$$

4.2 Transforming Metric Perturbations

Eqs. (4.6)–(4.8) (or for post-Newtonian theory, Eqs. (4.14)–(4.16)) can be used to study the way the metric perturbations change under a gauge transformation.

Using Eq. (4.14), ϕ transforms as follows:

$$\begin{aligned}
\tilde{h}_{00} &= h_{00} + g_{00,\mu}\xi^{\mu} + 2g_{0\mu}\xi'^{\mu} \\
&= h_{00} + g'_{00}\xi^{0} + g_{00,i}\xi^{i} + 2g_{00}\xi'^{0} + 2g_{0i}\xi'^{i} \\
-2a^2\tilde{\phi} &= -2a^2\phi - 2aa'(1+\phi)\xi^{0} - 2a^2\phi'\xi^{0} - 2a^2\phi_{,i}\xi^{i} \\
&\quad - 2a^2(1+\phi)\xi'^{0} + a^2(B_{,i} - S_{,i})\xi'^{i} \\
\tilde{\phi} &= \phi + \mathcal{H}\xi^{0} + \phi\xi^{0} + \phi'\xi^{0} + \phi_{,i}\xi^{i} + \xi'^{0} + \phi\xi'^{0} + (B_{,i} - S_{i})\xi'^{i}. \qquad (4.17)
\end{aligned}$$

In cosmological perturbation theory, at linear order, this reduces to

$$\tilde{\phi} = \phi + \mathcal{H}\xi^{0} + \xi'^{0} + O(\varepsilon^2). \qquad (4.18)$$

Whereas in post-Newtonian theory, keeping terms up to order η^4,

$$\tilde{\phi} = \phi + \mathcal{H}\xi^{0} + \xi'^{0} + \phi_{,i}\xi^{i} + O(\eta^5). \qquad (4.19)$$

Note that $\mathcal{H}$ is of order η as it involves a conformal time derivative.

An important difference between the two approaches is that in post-Newtonian theory, at leading order, ϕ is gauge invariant (that is, one cannot change coordinates into a coordinate system where ϕ takes on a desirable value, say, zero, while remaining perturbatively close to the FLRW background), which is not the case in cosmological perturbation theory. It will be seen later (Section 4.2.1) that this has consequences for the viability of certain gauges in post-Newtonian theory.

The other components of the metric perturbation can be transformed in a similar manner. From Eq. (4.7),

$$\tilde{h}_{0i} = h_{0i} + g_{00}\xi^{0}{}_{,i} + g_{ij}\xi'^{j}. \qquad (4.20)$$

Since $\xi^{0}{}_{,i}$ and ξ'^{j} are both of order η^3 and the metric perturbations are of at most order η^2, the contribution of the metric perturbation in the two terms involving the metric is negligible, so decomposing this using Eq. (3.18),

$$a^2\left(\tilde{B}_{,i} - \tilde{S}_{i}\right) = a^2(B_{,i} - S_{i}) - a^2\xi^{0}{}_{,i} + a^2\delta_{ij}\xi'^{j}. \qquad (4.21)$$

Using the Helmholtz theorem, ξ_i can be decomposed into the derivative of a scalar ζ, and a divergenceless 3-vector ζ_i, such that

$$\xi_i = \zeta_{,i} + \zeta_i. \qquad (4.22)$$

This allows the change in h_{0i} due to the gauge transformation to be split into the change in the $B_{,i}$ (the derivative of a scalar) and the change in S_i (a divergenceless vector field), resulting in

$$\tilde{B}_{,i} - \tilde{S}_{i} = (B_{,i} - \xi^{0}{}_{,i} + \zeta'{}_{,i}) - (S_{i} - \zeta'{}_{i}). \qquad (4.23)$$

From Eq. (4.8),

$$\tilde{h}_{ij} = h_{ij} + 2g_{k(i}\xi^{k}{}_{,j)}. \tag{4.24}$$

Unlike above, the metric perturbations will give order η^4 terms, but these will be ignored as they are not important for any of the gauges under consideration. To order η^3,

$$-\tilde{\psi}\delta_{ij} + \tilde{E}_{,ij} + \tilde{F}_{(i,j)} + \frac{1}{2}\tilde{D}_{ij} = -\psi\delta_{ij} + (E_{,ij} + \zeta_{,ij}) + (F_{(i,j)} + \zeta_{(i,j)}) + \frac{1}{2}D_{ij}. \tag{4.25}$$

Combining all these results, in cosmological perturbation theory to leading order,

$$\tilde{\phi} = \phi + \mathcal{H}\xi^0 + \xi'^0 \tag{4.26}$$
$$\tilde{\psi} = \psi - \mathcal{H}\xi^0 \tag{4.27}$$
$$\tilde{B} = B + \zeta' - \xi^0 \tag{4.28}$$
$$\tilde{S}_i = S_i - \zeta'_i \tag{4.29}$$
$$\tilde{E} = E + \zeta \tag{4.30}$$
$$\tilde{F}_i = F_i + \zeta_i \tag{4.31}$$
$$\tilde{D}_{ij} = D_{ij} \tag{4.32}$$

In post-Newtonian theory,

$$\tilde{\phi} = \phi + \mathcal{H}\xi^0 + \xi'^0 + \phi_{,i}\xi^i \tag{4.33}$$
$$\tilde{\psi} = \psi \tag{4.34}$$
$$\tilde{B} = B + \zeta' - \xi^0 \tag{4.35}$$
$$\tilde{S}_i = S_i - \zeta'_i \tag{4.36}$$
$$\tilde{E} = E + \zeta \tag{4.37}$$
$$\tilde{F}_i = F_i + \zeta_i \tag{4.38}$$
$$\tilde{D}_{ij} = D_{ij} \tag{4.39}$$

where $\zeta_{,i} + \zeta_i = \xi_i$, and ζ^i is divergenceless. Since ξ^μ directly describes the change of the coordinates in Eq. (4.3), clearly the 3-velocity changes as

$$v^i \to \tilde{v}^i = v^i + \xi'^i. \tag{4.40}$$

Notably, this means that v^i cannot be set to zero by a gauge transformation in post-Newtonian theory, as $v^i \sim \eta$ but $\xi'^i \sim \eta^3$. This has consequences for several gauges, including the N-body gauge discussed below. Next, we repeat the analysis in [15] for two gauges discussed in that paper. First, we consider the synchronous gauge, because it is a popular gauge choice and a simple example, then the N-body gauge, as it has some similarities to a Machian gauge discussed in Section 4.3. In that section we discuss the Machian gauges of [17], determining whether they are viable in post-Newtonian theory.

4.2.1 Synchronous Gauge

The synchronous gauge is defined by the conditions

$$\phi = B = S_i = 0. \tag{4.41}$$

This leaves only spatial perturbations, so the perturbed metric has the line element

$$ds^2 = a^2(-d\tau^2 + (\delta^{ij} + h^{ij})dx^i dx^j). \tag{4.42}$$

This gauge is popular in numerical studies, and has a time coordinate $dt = ad\tau$ corresponding to the proper time of comoving observers that have fixed spatial coordinates.

This gauge choice can be achieved in cosmological perturbation theory by choosing ξ^μ such that $\xi'^0 + \mathcal{H}\xi^0 + \phi = 0$ (Eq. (4.26)) and $B_{,i} + S_i + \xi'_i - \xi^0 = 0$ (Eqs. (4.28)–(4.29)).

From Eq. (4.19), setting $\tilde{\phi}$ to zero in post-Newtonian theory requires finding a gauge generator ξ such that

$$\underbrace{\mathcal{H}\xi^0}_{O(\eta^4)} + \underbrace{\xi'^0}_{O(\eta^4)} + \underbrace{\phi_{,i}\xi^i}_{O(\eta^4)} + O(\eta^5) = \underbrace{-\phi}_{O(\eta^2)}. \tag{4.43}$$

No gauge generator that satisfies the smallness conditions will be able to satisfy Eq. (4.43), so the synchronous gauge cannot be realised in post-Newtonian theory. This is not a problem in cosmological perturbation theory because all quantities are of the same order of smallness ε.

4.2.2 N-Body Gauge

The N-body gauge is defined by two conditions. The first condition is

$$v + B = 0, \tag{4.44}$$

where v is the scalar velocity potential obtained by decomposing the 3-velocity as $v^i = \delta^{ij}v_{,i} + v^i_{\text{vec}}$. Since $\tilde{v}_i = v_i + \xi'_i = v_i + \zeta'_{,i} + \zeta'_i$, under a gauge transformation $v \to \tilde{v} = v + \zeta'$, so combined with Eq. (4.28),

$$\tilde{v} + \tilde{B} = v + B + 2\zeta' - \xi^0. \tag{4.45}$$

This condition can be easily achieved in cosmological perturbation theory by picking $\xi^0 = v + B$ (for example), but in post-Newtonian theory v is much larger than the other quantities—it is of order η while the other three quantities are of order η^3. As such, it is too large for any choice of small gauge generator to reduce to zero, and so the gauge is not viable in post-Newtonian theory.

The second condition for the N-body gauge is[1]

$$\psi - \frac{1}{3}\nabla^2 E = 0, \tag{4.46}$$

where the left-hand side gives the scalar deformation of the spatial volume. Under a gauge transformation, this becomes

$$\psi - \frac{1}{3}\nabla^2(E + \zeta) = 0, \tag{4.47}$$

so this gauge can be achieved by picking ζ to solve $\nabla^2\zeta = 3\psi - \nabla^2 E$. While the other condition cannot be achieved in post-Newtonian theory, this one can be. The Mach 2 gauge

[1] There is a sign error in the corresponding equation in [15] which carries through the rest of their analysis but does not significantly affect their results.

in the following section makes use of this condition, and it may be possible to develop variant gauges that use an alternative condition on the time component of the gauge generator while still retaining some of the useful properties of the N-body gauge. For example, [29] uses the time slicing of the Poisson gauge combined with Eq. (4.46)'s spatial condition.

4.3 Machian Gauges

Bičák, Katz, and Lynden-Bell [17] consider three gauges that best embody Mach's principle, as discussed in the Chapter 1. They describe these gauges in the following way: "In these gauges local inertial frames can be determined instantaneously via the perturbed Einstein field equations from the distributions of energy and momentum in the universe." Each of these gauges has a different gauge condition on the time slicing, but all of them share a common set of three conditions on the spatial metric on the slices.

Smarr and York (1978) [30] treat full general relativity as an evolution of initial Cauchy data given on a spatial hypersurface. They require a "minimal-distortion" shift vector, because this allows kinematic and dynamical effects to be suitably separated in order to study the kinematics of observers threading the hypersurfaces. Having a minimal-distortion shift vector is equivalent to the three ($k = 1, 2, 3$) conditions

$$\nabla_\ell {h_T}^\ell{}_k = 0, \tag{4.48}$$

where h_{Tij} is the traceless part of h_{ij}. Following Smarr and York, Bičák *et al.* add this condition to each of their Machian gauge conditions. In terms of the Helmholtz decomposition, $\psi\delta_{ij} + 2E_{,ij}$ is the trace part of h_{ij}, so the remaining terms give the traceless part:

$$h_{Tij} = 2F_{i,j} + D_{ij}. \tag{4.49}$$

This implies that

$$\nabla_\ell {h_T}^\ell{}_k = 2F^\ell{}_{,k} + D^\ell{}_k = 0 \tag{4.50}$$

as F_i and D_{ij} are divergenceless, so this condition is automatically satisfied in this formalism (both post-Newtonian theory and cosmological perturbation theory). With this condition, Bičák *et al.* define three gauges, which they call the Mach 1, Mach 2, and Mach 3 gauges. They investigate these gauges from the point of view of cosmological perturbation theory; here we use the post-Newtonian approach.

4.3.1 Mach 1: Uniform Hubble Expansion Gauge

The uniform Hubble expansion gauge, considered by Bardeen [16], is defined by the condition

$$\delta\theta = 0, \tag{4.51}$$

that is, at every point on a hypersurface, the expansion θ is the same as the expansion of the background. As $\mathcal{K} = \delta\theta$, this means that the scalar extrinsic curvature of constant τ hypersurfaces is the same as in the unperturbed FLRW background. Since extrinsic curvature is constant on these unperturbed hypersurfaces, this gauge is sometimes called the constant mean extrinsic curvature (CMEC or CMC) gauge.

If the condition in Eq. (4.48) is added, the resulting gauge is called the *Mach 1 gauge*.

To see whether this condition is achievable in post-Newtonian theory, the transformation of $\nabla_\mu n^\mu$ in Eq. (3.40) under an arbitrary gauge transformation can be studied.

$$\begin{aligned}\theta = \nabla_\mu n^\mu \to \frac{1}{a}\Big[&\partial_k v^k + 3\mathcal{H}\Big(1 - \phi + \frac{v^2}{2}\Big) - v^k \partial_k \big(3\psi - \nabla^2 E + \nabla^2 \zeta\big) \\ &+ \partial_0 (3\psi - \nabla^2 E + \nabla^2 \zeta) + \phi_{,k} v^k + {v_k}' v^k\Big] + O(\eta^5).\end{aligned} \tag{4.52}$$

Recalling that $\bar{\theta} = 3\mathcal{H}/a$, the remaining terms make up $\delta\theta$. It is immediately clear that $\delta\theta$ cannot be set to zero by a gauge choice, because ζ is of order η^2, while there is a term of order η, namely $a^{-1}\partial_k v^k$.

As a weaker condition, it may be useful to be able to set the terms of order η^3 to zero (there are no terms of order η^2 or η^4). This would result in

$$\theta = \bar{\theta} + \frac{1}{a}\partial_k v^k, \tag{4.53}$$

so that the expansion would be equal to the background expansion plus the spatial divergence of the fluid 3-velocity in the hypersurface. This choice can be justified on physical grounds. Operationally, one cannot distinguish between the divergence of the velocity field and an isotropic component of the local expansion. For this reason, the condition in Eq. (4.53) means that the observed expansion will be uniform regardless of what is considered to be background expansion and what is considered the divergence $\partial_k v^k$ with respect to the background, which is arguably a more physically motivated choice.

To determine whether this gauge choice is possible, it is helpful to collect some of the terms together to more easily identify the form of the resulting differential equations. The terms of order η^3 can be written as

$$v^k(\partial_k(X+Y) + Z) + \partial_0(X+Y) + W = 0, \tag{4.54}$$

where $X = \nabla^2\zeta$ is the variable to be solved for, $Y = 3\psi - \nabla^2 E$, $W = 3\mathcal{H}/a\phi$, and $v^k Z$ consists of the remaining two terms of order η^3. To solve this, it is necessary to be able to solve the following four equations simultaneously:

$$0 = \partial_0(X+Y) + W \tag{4.55}$$

$$0 = \partial_k(X+Y) + Z \qquad k = 1,2,3. \tag{4.56}$$

Differentiating the first equation with respect to a spatial coordinate, and the second with respect to time shows the impossibility of this under general conditions:

$$0 = \partial_0\partial_k(X+Y) + \partial_k W \tag{4.57}$$

$$0 = \partial_0\partial_k(X+Y) + \partial_0 Z. \tag{4.58}$$

This would require that

$$\partial_0(\phi_{,k} + {v_k}') = \frac{3\mathcal{H}}{a}\phi_{,k}, \tag{4.59}$$

which cannot be set by choice of gauge, so this gauge is non-viable in post-Newtonian theory.

4.3.2 Mach 2: Uniform Scalar Curvature Gauge

Another geometric property of spatial slices is the spatial intrinsic scalar curvature. Like other quantities, it can be split into a background part and a perturbed part,

$$ {}^3\mathcal{R} = {}^3\bar{\mathcal{R}} + \delta\mathcal{R}. \tag{4.60} $$

The *Mach 2 gauge* is given by Eq. (4.48) plus the uniform spatial intrinsic curvature condition:

$$ \delta\mathcal{R} = 0. \tag{4.61} $$

The FLRW metric being used is spatially flat, so ${}^3\bar{\mathcal{R}} = 0$, so in this case the condition is equivalent to ${}^3\mathcal{R} = 0$.

The stronger condition

$$ h^n{}_n = 0 \tag{4.62} $$

is called the *Mach 2* gauge*. In post-Newtonian theory, $h^i{}_i = a^2(-6\psi + 2E^{,i}{}_i)$ so this gauge condition is equivalent to the condition

$$ 3\psi - \nabla^2 E = 0. \tag{4.63} $$

This condition was given previously in Eq. (4.46) as the second condition for the N-body gauge, which was achievable in post-Newtonian theory through the choice $\nabla^2\zeta = -3\psi + \nabla^2 E$.

4.3.3 Mach 3: Minimal-Shear Hypersurfaces Gauge

The *Mach 3 gauge* is given by Eq. (4.48) with the condition

$$ \nabla_i \nabla_j \sigma^{ij} = 0, \tag{4.64} $$

called the minimal-shear hypersurface condition. This condition was discussed in [16] and [17], in the equivalent form

$$ K_{ij} - \frac{1}{3}\mathcal{K}\delta^{ij} = 0. \tag{4.65} $$

The stronger condition

$$ \nabla_k h^k{}_0 = 0 \tag{4.66} $$

defines the *Mach 3* gauge*. This gauge is also known as the Poisson gauge. In terms of the Helmholtz decomposition quantities, this condition can be written as

$$ \nabla_k h^k{}_0 = \nabla_k (B^{,k} - S^k) = 0. \tag{4.67} $$

B and S_i each transform the same way in post-Newtonian theory as they do in cosmological perturbation theory, so under an arbitrary gauge transformation,

$$ \begin{aligned} \nabla_k(\tilde{B}^{,k} - \tilde{S}^k) &= \nabla_k((B^{,k} + \zeta'^{,k} - \xi^{0,k}) - (S^k - \zeta'^k)) && (4.68) \\ &= \nabla_k(B^{,k} - S^k + \xi^{0,k} + \xi'^k). && (4.69) \end{aligned} $$

This can be set to zero by choosing, for example, $\xi^0 = -B$, $\xi'^i = S^i$, so this gauge is viable in both approaches. Note that this allows an even stronger condition, $h^k{}_0 = 0$.

Chapter 5

Conclusion

We have reviewed the recent work of Clifton *et al.* [15], discussing cosmological perturbation theory and post-Newtonian theory. This involved discussing perturbation theory and the gauge problem, and studying how metric perturbations transform under a gauge transformation in both of the approaches to perturbation theory. Following this, we repeated the analysis done by Clifton *et al.* for the synchronous gauge and the N-body gauge.

In Section 4.3, we performed a similar analysis of three gauges proposed by Bičák, Katz, and Lynden-Bell [17]. These gauges embody Mach's principle because in the sense that instantaneous local inertial frames are determined from only the distribution of energy and momentum in the universe. The first of these Machian gauges is the uniform Hubble expansion gauge, in which spatial hypersurfaces have constant scalar extrinsic curvature (or equivalently, the expansion of a congruence of curves normal to these hypersurfaces has a uniform expansion rate). To investigate this, an expression for the extrinsic curvature in the post-Newtonian formalism was derived in Section 3.4.1 and its change under a general gauge transformation was studied in Section 4.3.1. We found that this gauge is in general nonviable in post-Newtonian theory.

The second Machian gauge is the uniform scalar curvature gauge, which is similar to the previous gauge except that it is the intrinsic scalar curvature of the hypersurfaces that is constant, rather than the extrinsic scalar curvature. We found that this gauge is viable in post-Newtonian theory. This gives a specific example of a gauge choice that is a variation of the N-body gauge with an alternative specification of the temporal condition, which Clifton *et al.* noted may be possible.

The third Machian gauge is the minimal-shear hypersurfaces gauge. The Poisson gauge condition is a stronger condition than the minimal-shear hypersurface condition, and consequently Bičák *et al.* call this gauge the Mach 3* gauge. It is possible to achieve this condition through a gauge transformation in post-Newtonian theory, so this gauge is another gauge viable in both perturbative approaches.

The viability of these two gauges is interesting for several reasons. Firstly, is it useful to have more viable post-Newtonian gauge choices for the purpose of simplifying calculations and interpreting physical problems. Additionally, it is interesting to find gauges that are viable in both approaches to perturbation theory, since most of the gauges investigated in [15] were found to be nonviable in post-Newtonian theory. Such gauges are necessary for studies that attempt to simultaneously model small-scale nonlinear structures and large-scale linear structures, such as two-parameter perturbation theory [9, 10] which combines post-Newtonian theory and cosmological perturbation theory, or numerical code for simulations such as GEVOLUTION [31].

As discussed in Chapter 1, post-Newtonian theory is relevant to numerical cosmological simulations, for the purpose of interpreting the results of simulations and providing relativistic corrections to Newtonian simulations by post-processing results. On account of this, understanding post-Newtonian theory and gauges viable within it will be crucial for maintaining accuracy in the increasingly precise numerical simulations. With rapid increases in the precision of observations, we are at a point where omitting relativistic effects may limit the possible predictive accuracy of Newtonian methods. The gauges that we have found to be viable may provide alternative starting points for implementing relativistic simulations.

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Appendix A

Post-Newtonian Christoffel Symbols

In this appendix, we compute the post-Newtonian Christoffel symbols including terms up to order η^4. Section 4.1 justifies truncating the results at this order specifically. Despite being calculated from the same general formula, the post-Newtonian Christoffel symbols are different to those in cosmological perturbation theory due to the fact that we are truncating them at a certain order, and some terms become larger or smaller relative to the other terms when changing from one approach to the other.

The general formula for Christoffel symbols is

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\big(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\big) \tag{A.1}$$

Using this, the four sets of Christoffel symbols in Eq. (2.19) can be computed. First,

$$\Gamma^{0}{}_{00} = \frac{1}{2} g^{0\lambda}(2\partial_0 g_{0\lambda} - \partial_\lambda g_{00}) \tag{A.2}$$

Splitting $g^{0\lambda}$ into $\bar{g}^{0\lambda} - h^{0\lambda}$, noting that the background metric is diagonal,

$$= \frac{1}{2}\bar{g}^{00}\partial_0 g_{00} - \frac{1}{2}h^{0\lambda}(2\partial_0 g_{0\lambda} - \partial_\lambda g_{00}) \tag{A.3}$$

$$= \frac{1}{2}\bar{g}^{00}\partial_0 g_{00} - \frac{1}{2}h^{00}(2\partial_0 g_{00} - \partial_0 g_{00}) - \frac{1}{2}h^{0k}(2\partial_0 g_{0k} - \partial_k g_{00}) \tag{A.4}$$

The background metric has no spatial dependence,

$$= \frac{1}{2}\bar{g}^{00}\partial_0 g_{00} - \frac{1}{2}h^{00}\partial_0 g_{00} - \frac{1}{2}h^{0k}(2\partial_0 h_{0k} - \partial_k h_{00}) \tag{A.5}$$

Since h^{0k} is of order η^3, the final term is $O(\eta^5)$.

$$= \frac{1}{2}\bar{g}^{00}\partial_0 g_{00} - \frac{1}{2}h^{00}\partial_0 \bar{g}_{00} + O(\eta^5) \tag{A.6}$$

$$= -\frac{1}{2a^2}\partial_0\big(-a^2 + h_{00}\big) - \frac{1}{2}h^{00}\partial_0 \bar{g}_{00} + O(\eta^5) \tag{A.7}$$

$$= \mathcal{H} - \frac{1}{2a^2}{h_{00}}' + a'ah^{00} + O(\eta^5) \tag{A.8}$$

So, in terms of the Helmholtz decomposition,

$$\Gamma^{0}{}_{00} = \mathcal{H} + \phi' + O(\eta^5) \tag{A.9}$$

Where $\mathcal{H} = \frac{a'}{a}$ is the conformal Hubble parameter. Similarly, for the other terms,

$$\Gamma^0{}_{0k} = \frac{1}{2} g^{0\lambda}(\partial_k g_{0\lambda} + \partial_0 g_{k\lambda} - \partial_\lambda g_{k0}) \tag{A.10}$$

$$= \frac{1}{2}\bar{g}^{00}(\partial_k g_{00}) - \frac{1}{2}h^{0\lambda}(\partial_k h_{0\lambda} + \partial_0 g_{k\lambda}) + O(\eta^5) \tag{A.11}$$

$$= \frac{1}{2}\bar{g}^{00}\partial_k h_{00} - \frac{1}{2}h^{00}\partial_k h_{00} - \frac{1}{2}h^{0j}\partial_0 \bar{g}_{jk} + O(\eta^5) \tag{A.12}$$

$$= -\frac{1}{2a^2}\partial_k h_{00} - \frac{1}{2}h^{00}\partial_k h_{00} - a' a h^{0k} + O(\eta^5) \tag{A.13}$$

$$= \phi_{,k} - 2\phi\phi_{,k} - \mathcal{H}(B_{,k} - S_k) + O(\eta^5) \tag{A.14}$$

Next,

$$\Gamma^k{}_{k0} = \frac{1}{2}\bar{g}^{kj}\partial_0 g_{kj} - \frac{1}{2}h^{k\lambda}(\partial_k h_{0\lambda} + \partial_0 g_{k\lambda}) + O(\eta^5) \tag{A.15}$$

$$= \frac{1}{2}\bar{g}^{kj}\partial_0 g_{kj} - \frac{1}{2}h^{kj}\partial_0 \bar{g}_{kj} + O(\eta^5) \tag{A.16}$$

$$= -3\mathcal{H} + 3\psi' - \delta^{jk}(E'{}_{,jk} + F'{}_{(j,k)}) + O(\eta^5) \tag{A.17}$$

$$= -3\mathcal{H} + 3\psi' - \nabla^2 E' + O(\eta^5) \tag{A.18}$$

as F_j and D_{ij} are divergenceless. Finally,

$$\Gamma^k{}_{kj} = \frac{1}{2}\bar{g}^{k\ell}\big(\partial_k h_{j\ell} + \partial_j h_{k\ell} - \partial_\ell h_{jk}\big) - \frac{1}{2}h^{k\lambda}\big(\partial_k g_{j\lambda} + \partial_j g_{k\lambda} - \partial_\lambda g_{jk}\big) \tag{A.19}$$

$$= \frac{1}{2}\bar{g}^{k\ell}\partial_j h_{k\ell} - \frac{1}{2}h^{k0}\partial_0 \bar{g}_{jk} - \frac{1}{2}h^{k\ell}\partial_j g_{k\lambda} + O(\eta^5) \tag{A.20}$$

$$= -3\psi_{,j} + \nabla^2 E_{,j} + \mathcal{H}(B_{,j} - S_j) - h^{kl}\partial_j h_{kl} + O(\eta^5) \tag{A.21}$$